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A Trigonometrical Sum and the Gibbs' Phenomenon in Fourier's Series.*

BY H. S. CARSLAW.

§ 1. If $f(x)$ is a function which can be expanded in a Fourier's Series in the interval $0 \leq x \leq 2\pi$, with a discontinuity at $x=a$ such that $f(a-0)$ and $f(a+0)$ exist, the sum of the Fourier's Series for $x=a$ is $\frac{1}{2}\{f(a+0) + f(a-0)\}$.

Denoting by $S_n(x)$ the sum of the terms up to and including those in $\sin nx$ and $\cos nx$, the curves $y=S_n(x)$ are usually spoken of as the approximation curves for the series. Up till 1899 it was believed that each approximation curve, for large values of n , passes at a steep gradient from a point near $(a, f(a-0))$ to a point near $(a, \frac{1}{2}(f(a+0) + f(a-0)))$, and then on to a point near $(a, f(a+0))$, afterwards oscillating about the curve $y=f(x)$ till another discontinuity of $f(x)$ is met.

In 1899 J. Willard Gibbs pointed out† that the approximation curves for the sine series

$$2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots)$$

do not behave in this way.

In the interval $0 \leq x \leq 2\pi$, the sum of this series is $f(x)$, where $f(x)$ is defined as follows:

$$\left. \begin{aligned} f(0) &= f(2\pi) = 0, \\ f(x) &= x, \quad 0 < x < \pi, \\ f(x) &= x - 2\pi, \quad \pi < x < 2\pi. \end{aligned} \right\}$$

Gibbs stated in effect that the curve $y=S_n(x)$ for this series, for large values of n , rises just before $x=\pi$ to a point very nearly at a height $2 \int_0^\pi \frac{\sin x}{x} dx$ above the axis of x , passes from that point through $(\pi, 0)$ at a steep gradient to a point very nearly at the same distance $2 \int_0^\pi \frac{\sin x}{x} dx$ below the axis, while in the rest of the interval $0 \leq x \leq 2\pi$ it oscillates above and below the curve $y=f(x)$.

* Communicated to the American Mathematical Society, October, 1915.

† *Nature*, Vol. LIX (1899), p. 606.

His statement was not accompanied by any proof. Though the remainder of the correspondence, of which his letter formed a part, attracted no little attention, this remarkable observation remained practically unnoticed for several years. In 1906 Bôcher returned to the subject in a memoir on Fourier's Series,* and extended Gibbs' results in two directions.

First of all he discussed the series

$$\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots,$$

which, in the interval $0 \leq x \leq 2\pi$, represents the function $f(x)$ defined as follows:

$$\left. \begin{aligned} f(0) &= f(2\pi) = 0, \\ f(x) &= \frac{1}{2}(\pi - x) \dots 0 < x < 2\pi. \end{aligned} \right\}$$

He showed that the approximation curve

$$y = \sin x + \frac{1}{2} \sin 2x + \dots + \frac{1}{n} \sin nx = S_n(x),$$

in the interval $0 \leq x \leq 2\pi$, and for large values of n , takes the form of a wavy curve which keeps crossing and recrossing the line $y = \frac{1}{2}(\pi - x)$ and reaches its greatest distance from that line (distances being measured parallel to the axis of y) at the points $x_1, x_2, \dots, x_{2n}$, where $x_r = \frac{2r\pi}{2n+1}$.

Further, these greatest distances in the r -th wave are, for large values of n , approximately equal to their limiting values

$$\frac{1}{2}\pi - \int_0^{r\pi} \frac{\sin x}{x} dx.$$

In the second place he showed that the phenomenon in question also appears in the Fourier's Series for a large class of functions. How large this class is will appear in the concluding section of this paper.

In Bôcher's own words:† If $S_n(x)$ denotes the sum of the Fourier's expansion of $f(x)$, the curve $y = S_n(x)$ will, for large values of n , pass in almost a vertical direction through a point whose abscissa is a and whose ordinate is almost $\frac{1}{2}\{f(a+0) + f(a-0)\}$. The curve then rises and falls abruptly on the two sides of this point to the neighborhood of the curve

* *Annals of Mathematics* (2), Vol. VII (1906). See also a recent paper in *Crelle's Journal*, Bd. 144 (1914), entitled "On Gibbs' Phenomenon."

Reference should also be made to Runge's "Theorie und Praxis der Reihen" (1904), pp. 170-180. A certain series is there discussed, and the nature of the jump in the approximation curve described. But no reference is made to Gibbs, and the example seems to have been regarded as quite an isolated one.

† *Loc. cit.*, *Annals of Mathematics*, p. 131. Both Theorems I and II, p. 131, should be noted.

$y=f(x)$, and oscillates about this curve, lying alternately above and below it. The highest (or lowest) point of the k -th wave to the right and left of a will, for large values of n , be approximately at the points

$$a \pm \frac{2k\pi}{2n+1},$$

and the height of these waves will be approximately

$$\frac{f(a+0)-f(a-0)}{\pi} P_k,$$

(where $P_k = \frac{1}{2}\pi - \int_0^{k\pi} \frac{\sin x}{x} dx$).

In the above sentence, a is a value of x at which $f(x)$ has an ordinary discontinuity, and Bôcher adds a footnote to the effect that "it must be borne in mind that we measure the height of a wave from the curve $y=f(x)$ in a direction parallel to the axis of y ."

§ 2. In the function studied by Bôcher, $\frac{1}{2}(\pi-x)$,

$$\begin{aligned} S_n(x) &= \sin x + \frac{1}{2}\sin 2x + \dots + \frac{1}{n}\sin nx \\ &= \int_0^x (\cos \alpha + \cos 2\alpha + \dots + \cos n\alpha) d\alpha \\ &= \frac{1}{2} \int_0^x \frac{\sin(n+\frac{1}{2})\alpha}{\sin \frac{1}{2}\alpha} d\alpha - \frac{1}{2}x. \end{aligned}$$

The properties of the maxima and minima of $S_n(x)$ are not so easy to obtain, nor are they so useful in this case as those of

$$\begin{aligned} R_n(x) &= \frac{1}{2}(\pi-x) - S_n(x) \\ &= \frac{1}{2}\pi - \frac{1}{2} \int_0^x \frac{\sin(n+\frac{1}{2})\alpha}{\sin \frac{1}{2}\alpha} d\alpha. \end{aligned}$$

It is the maxima and minima of $R_n(x)$ with which Bôcher deals in his memoir.

Gronwall has discussed* the somewhat complicated properties of the maxima and minima of $S_n(x)$ for this series, and deduced Gibbs' Phenomenon for the first wave, and the general case of Fourier's Series.

In this paper I use the series

$$2(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots),$$

* *Mathematische Annalen*, Bd. LXXII (1912). Some of Gronwall's results had been published a few months earlier by Dunham Jackson in a paper in the *Rendiconti di Circolo Matematico di Palermo*, T. XXXII (1911).

which, in the interval $-\pi \leq x \leq \pi$,* represents the function $f(x)$ defined by the equations

$$\left. \begin{aligned} f(-\pi) &= f(0) = f(\pi) = 0, \\ f(x) &= -\frac{1}{2}\pi \dots -\pi < x < 0, \\ f(x) &= \frac{1}{2}\pi \dots 0 < x < \pi. \end{aligned} \right\}$$

I take $S_n(x)$ for the sum of the first n terms of this series, so that

$$S_n(x) = 2 \left(\sin x + \frac{1}{3} \sin 3x + \dots + \frac{1}{2n-1} \sin (2n-1)x \right).$$

A number of interesting properties of the turning points of $y = S_n(x)$ are obtained by quite simple methods; and all the features of Gibbs' Phenomenon for this series follow immediately from these properties.

In the concluding article the extension to the general case of Fourier's Series is given; but the method does not differ materially from that of Bôcher.

It is, of course, not an unusual thing that the curves $y = S_n(x)$ for a series, whose terms are continuous functions, differ considerably, even for large values of n , from the curve $y = \lim_{n \rightarrow \infty} (S_n(x)) = S(x)$.

They certainly do so in the neighborhood of a point where $S(x)$, the sum of the series, is discontinuous. And they also may do so in the neighborhood of a point where $S(x)$ is continuous.†

Ex. (i). If

$$S_n(x) = \frac{nx}{1+n^2x^2}, \quad x \geq 0,$$

we have

$$S(x) = 0, \quad x \geq 0.$$

In this case $S_n(x)$ has a maximum value $\frac{1}{2}$ at $x = \frac{1}{n}$. As n gets larger and larger, this summit is pushed towards $x = 0$ but its height remains equal to $\frac{1}{2}$.

Ex. (ii). If

$$S_n(x) = \frac{n^2x}{1+n^3x^2}, \quad x \geq 0,$$

we have

$$S(x) = 0, \quad x \geq 0.$$

In this case $S_n(x)$ has a maximum value $\frac{1}{2}n^{\frac{1}{3}}$ at $x = \frac{1}{n^{\frac{2}{3}}}$. As n gets larger and larger, this summit is pushed towards $x = 0$ but its height increases without limit.

* It is more convenient for my purpose to take the interval $-\pi \leq x \leq \pi$ than $0 \leq x \leq 2\pi$. This does not affect the argument to any extent.

† Cf. Osgood, *Bulletin of the American Math. Soc.* (2), Vol. III (1897), p. 63; AMERICAN JOURNAL OF MATHEMATICS, Vol. XIX (1897), p. 155; Hobson, "Theory of Functions of a Real Variable," p. 480; Bromwich, "Theory of Infinite Series," p. 110; Carslaw, "Fourier's Series and Integrals," Ch. III.

The existence of maxima (or minima) of $S_n(x)$, the abscissæ of which tend towards a (a being a value of x for which the sum of the series is discontinuous), while their ordinates remain at a finite distance from $S(a+0)$ and $S(a-0)$, as n increases, is the chief feature of the Gibbs' Phenomenon in Fourier's Series. And it is most remarkable that its occurrence in Fourier's Series remained undiscovered till so recent a date.*

THE TRIGONOMETRICAL SUM

$$S_n(x) = 2 \left(\sin x + \frac{1}{3} \sin 3x + \dots + \frac{1}{2n-1} \sin (2n-1)x \right).$$

§ 3. If we define $f(x)$ by the equations

$$\left. \begin{aligned} f(-\pi) &= f(0) = f(\pi), \\ f(x) &= \frac{1}{2}\pi \dots 0 < x < \pi, \\ f(x) &= -\frac{1}{2}\pi \dots -\pi < x < 0, \end{aligned} \right\}$$

the Fourier's Series for $f(x)$ in the interval $-\pi \leq x \leq \pi$ is

$$2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots).$$

We denote by $S_n(x)$ this sum up to and including the term in $\sin(2n-1)x$. Then

$$S_n(x) = 2 \int_0^x (\cos x + \cos 3x + \dots + \cos(2n-1)x) dx = \int_0^x \frac{\sin 2n\alpha}{\sin \alpha} d\alpha.$$

Since $S_n(x)$ is an *odd* function, we need only consider the interval $0 \leq x \leq \pi$.

We proceed to obtain the properties of the maxima and minima of this function $S_n(x)$.

I. Since, for any integer m , $\sin(2m-1)(\frac{1}{2}\pi + x') = \sin(2m-1)(\frac{1}{2}\pi - x')$, it follows from the series that $S_n(x)$ is *symmetrical* about $x = \frac{1}{2}\pi$, and when $x=0$ and $x=\pi$ it is zero.

II. When $0 < x < \pi$, $S_n(x)$ is *positive*.

From (I) we need only consider $0 < x \leq \frac{1}{2}\pi$. We have

$$S_n(x) = \int_0^x \frac{\sin 2n\alpha}{\sin \alpha} d\alpha = \frac{1}{2n} \int_0^{2nx} \frac{\sin \alpha}{\sin \frac{\alpha}{2n}} d\alpha, \quad 0 < x \leq \frac{1}{2}\pi.$$

The denominator in the integrand is positive and continually increases in the

* Cf. also Weyl: (i) "Die Gibbs'sche Erscheinung in der Theorie der Kugel-Funktionen," *Rend. Circ. Mat. d. Palermo*, T. XXIX (1910); (ii) "Über die Gibbs'sche Erscheinung und verwandte Konvergenzphänomene," *ibid.*, T. XXX (1910).

interval of integration. By considering the successive waves in the graph of $\sin \alpha \operatorname{cosec} \frac{\alpha}{2n}$, the last of which may or may not be completed, it is clear that the integral is positive.

III. *The turning points of $y=S_n(x)$ are given by*

$$\begin{cases} x_1 = \frac{\pi}{2n}, x_3 = \frac{3\pi}{2n}, \dots, x_{2n-1} = \frac{2n-1}{2n}\pi \text{ (maxima),} \\ x_2 = \frac{\pi}{n}, x_4 = \frac{2\pi}{n}, \dots, x_{2(n-1)} = \frac{n-1}{n}\pi \text{ (minima).} \end{cases}$$

We have

$$y = \int_0^x \frac{\sin 2n\alpha}{\sin \alpha} d\alpha, \text{ and } \frac{dy}{dx} = \frac{\sin 2nx}{\sin x}.$$

The result follows at once.

IV. *As we proceed from $x=0$ to $x=\frac{1}{2}\pi$, the heights of the maxima continually diminish, and the heights of the minima continually increase, n being kept fixed.*

Consider two consecutive maxima in the interval $0 < x \leq \frac{1}{2}\pi$, namely, $S_n\left(\frac{2m-1}{2n}\pi\right)$ and $S_n\left(\frac{2m+1}{2n}\pi\right)$, m being a positive integer less than or equal to $\frac{1}{2}(n-1)$. We have

$$\begin{aligned} S_n\left(\frac{2m-1}{2n}\pi\right) - S_n\left(\frac{2m+1}{2n}\pi\right) &= \frac{1}{2n} \int_{(2m+1)\pi}^{(2m-1)\pi} \frac{\sin \alpha}{\sin \frac{\alpha}{2n}} d\alpha \\ &= -\frac{1}{2n} \left\{ \int_{(2m-1)\pi}^{2m\pi} \frac{\sin \alpha}{\sin \frac{\alpha}{2n}} d\alpha + \int_{2m\pi}^{(2m+1)\pi} \frac{\sin \alpha}{\sin \frac{\alpha}{2n}} d\alpha \right\}. \end{aligned}$$

The denominator in both integrands is positive and it continually increases in the interval $(2m-1)\pi \leq \alpha \leq (2m+1)\pi$; also the numerator in the first is continually negative and in the second continually positive; the absolute values for elements at equal distances from $(2m-1)\pi$ and $2m\pi$ being the same.

Thus the result follows. Similarly for the minima, we have to examine the sign of

$$S_n\left(\frac{m-1}{n}\pi\right) - S_n\left(\frac{m}{n}\pi\right),$$

where m is a positive integer less than or equal to $\frac{1}{2}n$.

V. The first maximum to the right of $x=0$ is at $x=\frac{\pi}{2n}$ and its height continually diminishes as n increases. When n tends to infinity, its limit is

$$\int_0^\pi \frac{\sin x}{x} dx.$$

We have

$$S_n\left(\frac{\pi}{2n}\right) = \int_0^{\frac{\pi}{2n}} \frac{\sin 2n\alpha}{\sin \alpha} d\alpha = \frac{1}{2n} \int_0^\pi \sin \alpha \operatorname{cosec} \frac{\alpha}{2n} d\alpha,$$

and

$$S_n\left(\frac{\pi}{2n}\right) - S_{n+1}\left(\frac{\pi}{2n+2}\right) = \int_0^\pi \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} \right) d\alpha.$$

Since $\alpha/\sin \alpha$ continually increases from 1 to ∞ , as α passes from 0 to π , it is clear that in the interval with which we have to deal

$$\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} > 0.$$

Thus

$$S_n\left(\frac{\pi}{2n}\right) - S_{n+1}\left(\frac{\pi}{2(n+1)}\right) > 0.$$

But, from (I), $S_n(x)$ is positive when $0 < x < \pi$.

It follows that $S_n\left(\frac{\pi}{2n}\right)$ tends to a limit as n tends to infinity.

The value of this limit can be obtained by the method used by Bôcher for the integral $\int_0^x \frac{\sin(n+\frac{1}{2})\alpha}{\sin \frac{1}{2}\alpha} d\alpha$.^{*} But it is readily obtained from the definition of the Definite Integral as the limit of a sum.

For we have

$$\begin{aligned} S_n\left(\frac{\pi}{2n}\right) &= 2\left(\frac{\pi}{2n}\right) \left\{ \frac{2n}{\pi} \sin \frac{\pi}{2n} + \frac{2n}{3\pi} \sin \frac{3\pi}{2n} + \dots + \frac{2n}{(2n-1)\pi} \sin \frac{2n-1}{2n} \pi \right\} \\ &= 2 \sum_{\substack{m=1 \\ 2nh=\pi}}^{2n} \left(\frac{\sin mh}{mh} h \right) - \sum_{\substack{m=1 \\ nh=\pi}}^n \left(\frac{\sin mh}{mh} h \right). \end{aligned}$$

Therefore

$$\text{Lt.}_{n \rightarrow \infty} S_n\left(\frac{\pi}{2n}\right) = 2 \int_0^\pi \frac{\sin x}{x} dx - \int_0^\pi \frac{\sin x}{x} dx = \int_0^\pi \frac{\sin x}{x} dx.$$

^{*} *Loc. cit.*, *Annals of Mathematics*, p. 124. Also Hobson, *loc. cit.*, p. 649.

VI. The result obtained in (V) for the first wave is a special case of the following:

The r -th maximum to the right of $x=0$ is at $x_{2r-1} = \frac{2r-1}{2n}\pi$, and its height continually diminishes as n increases, r being kept constant. When n tends to infinity, its limit is $\int_0^{(2r-1)\pi} \frac{\sin x}{x} dx$, which is greater than $\frac{1}{2}\pi$.

The r -th minimum to the right of $x=0$ is at $x_{2r} = \frac{r}{n}\pi$, and its height continually increases as n increases, r being kept constant. When n tends to infinity, its limit is $\int_0^{2r\pi} \frac{\sin x}{x} dx$, which is less than $\frac{1}{2}\pi$.

To prove these theorems we consider first the integral

$$\int_0^{m\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} \right) d\alpha,$$

m being a positive integer less than or equal to $2n-1$, so that $0 < \frac{\alpha}{2n} < \pi$ in the interval of integration.

Then

$$F(\alpha) = \frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} > 0$$

in this interval. (Cf. (V)).

Further,

$$\begin{aligned} F'(\alpha) &= \frac{1}{(2n+2)^2} \cos \frac{\alpha}{2n+2} \operatorname{cosec}^2 \frac{\alpha}{2n+2} - \frac{1}{(2n)^2} \cos \frac{\alpha}{2n} \operatorname{cosec}^2 \frac{\alpha}{2n} \\ &= \alpha^{-2} \{ \phi^2 \cos \phi \operatorname{cosec}^2 \phi - \psi^2 \cos \psi \operatorname{cosec}^2 \psi \}, \end{aligned}$$

where $\phi = \alpha/(2n+2)$ and $\psi = \alpha/2n$.

But

$$\frac{d}{d\phi} (\phi^2 \cos \phi \operatorname{cosec}^2 \phi) = -\phi \operatorname{cosec}^3 \phi [\phi(1 \mp \cos \phi)^2 \pm 2 \cos \phi (\phi \mp \sin \phi)].$$

And the right-hand side of the equation will be seen to be negative, choosing the upper signs for $0 < \phi < \frac{1}{2}\pi$ and the lower for $\frac{1}{2}\pi < \phi < \pi$.

Therefore $\phi^2 \cos \phi \operatorname{cosec}^2 \phi$ diminishes as ϕ increases from 0 to π .

It follows, from the expression for $F(\alpha)$, that $F'(\alpha) > 0$, and $F(\alpha)$ increases with α in the interval of integration.

The curve

$$y = \sin x \left(\frac{1}{2n} \operatorname{cosec} \frac{x}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{x}{2n+2} \right), \dots, 0 < x < m\pi,$$

thus consists of a succession of waves of length π , alternately above and below the axis, and the absolute values of the ordinates at points at the same distance from the beginning of each wave continually increase.

It follows that, when m is equal to $2, 4, \dots, 2(n-1)$, the integral

$$\int_0^{m\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} \right) d\alpha$$

is negative; and, when m is equal to $1, 3, \dots, 2n-1$, this integral is positive.

Returning to the maxima and minima, we have, for the r -th maximum to the right of $x=0$,

$$\begin{aligned} S_n(x_{2r-1}) - S_{n+1}(x_{2r-1}) &= \int_0^{\frac{2r-1}{2n}\pi} \frac{\sin 2n\alpha}{\sin \alpha} d\alpha - \int_0^{\frac{2r-1}{2(n+1)}\pi} \frac{\sin 2(n+1)\alpha}{\sin \alpha} d\alpha \\ &= \int_0^{(2r-1)\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} \right) d\alpha. \end{aligned}$$

Therefore from the above argument, $S_n(x_{2r-1}) > S_{n+1}(x_{2r-1})$.

Also for the r -th minimum to the right of $x=0$, we have

$$S_n(x_{2r}) - S_{n+1}(x_{2r}) = \int_0^{2r\pi} \sin \alpha \left(\frac{1}{2n} \operatorname{cosec} \frac{\alpha}{2n} - \frac{1}{2n+2} \operatorname{cosec} \frac{\alpha}{2n+2} \right) d\alpha,$$

and $S_n(x_{2r}) < S_{n+1}(x_{2r})$.

By an argument similar to that at the close of (V) we have

$$\operatorname{Lt.}_{n \rightarrow \infty} S_n(x_r) = \int_0^{r\pi} \frac{\sin x}{x} dx.*$$

It is clear that these limiting values are all greater than $\frac{1}{2}\pi$ for the maxima, and positive and less than $\frac{1}{2}\pi$ for the minima.

THE GIBBS' PHENOMENON FOR THE SERIES

$$2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots).$$

§ 4. From the Theorems (I)–(VI) of § 3 all the features of the Gibbs' Phenomenon for the series

$$2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots) \dots - \pi \leq x \leq \pi$$

follow immediately.

* For the values of $\int_0^{r\pi} \frac{\sin x}{x} dx$, see Bôcher, *loc. cit.*, *Annals of Mathematics*, p. 129.

It is obvious that we need only examine the interval $0 \leq x \leq \pi$, and that the discontinuity occurs at $x=0$.

For large values of n , the curve

$$y = S_n(x),$$

where $S_n(x) = 2 \left(\sin x + \frac{1}{3} \sin 3x + \dots + \frac{1}{2n-1} \sin (2n-1)x \right)$, rises at a steep gradient from the origin to its first maximum, which is very near, but above, the point $\left(0, \int_0^\pi \frac{\sin x}{x} dx\right)$ § 3, V}. The curve, then, falls at a steep gradient, without reaching the axis of x § 3, II}, to its first minimum, which is very near, but below, the point $\left(0, \int_0^{2\pi} \frac{\sin x}{x} dx\right)$ § 3, VI}. It then oscillates above and below the line $y = \frac{1}{2}\pi$, the heights (and depths) of the waves continually diminishing as we proceed from $x=0$ to $x = \frac{1}{2}\pi$ § 3, IV}; and from $x = \frac{1}{2}\pi$ to $x = \pi$, the procedure is reversed, the curve in the interval $0 \leq x \leq \pi$ being symmetrical about $x = \frac{1}{2}\pi$ § 3, I}.

The highest (or lowest) point of the r -th wave to the right of $x=0$ will, for large values of n , be at a point whose abscissa is $\frac{r\pi}{2n}$ § 3, III} and whose ordinate is very nearly $\int_0^{r\pi} \frac{\sin x}{x} dx$ § 3, VI}.

By increasing n the curve for $0 \leq x \leq \pi$ can be brought as close as we please to the lines

$$\left. \begin{aligned} x=0, & \quad 0 < y < \int_0^\pi \frac{\sin x}{x} dx, \\ 0 < x < \pi, & \quad y = \frac{\pi}{2}, \\ x=\pi, & \quad 0 < y < \int_0^\pi \frac{\sin x}{x} dx. \end{aligned} \right\}$$

We may state these results more definitely as follows:

(i) If ε is any positive number, as small as we please, there is a positive integer ν' such that

$$|\tfrac{1}{2}\pi - S_n(x)| < \varepsilon \quad \text{for } n \geq \nu', \quad \varepsilon \leq x \leq \tfrac{1}{2}\pi.$$

This follows from the uniform convergence of the Fourier's Series for $f(x)$ in an interval which does not include a discontinuity of $f(x)$.

(ii) Since the height of the first maximum to the right of $x=0$ tends

from above to $\int_0^\pi \frac{\sin x}{x} dx$ as n tends to infinity, there is a positive integer ν'' such that

$$0 < S_n\left(\frac{\pi}{2n}\right) - \int_0^\pi \frac{\sin x}{x} dx < \varepsilon \text{ for } n \geq \nu''.$$

(iii) Let ν''' be the integer next greater than $\frac{\pi}{2\varepsilon}$. Then the abscissa of the first maximum to the right of $x=0$, for $n \geq \nu'''$, is less than ε .

It follows from (i), (ii) and (iii) that, if ν is the greatest of the positive integers ν' , ν'' and ν''' , the curve $y=S_n(x)$, for $n \geq \nu$, behaves as follows:

It rises at a steep gradient from the origin to its first maximum, which is above $\int_0^\pi \frac{\sin x}{x} dx$ and within the rectangle

$$0 < x < \varepsilon, \quad 0 < y < \int_0^\pi \frac{\sin x}{x} dx + \varepsilon.$$

After leaving this rectangle, in which there may be many oscillations about $y = \frac{1}{2}\pi$, it remains within the rectangle $\varepsilon < x < \pi - \varepsilon$, $\frac{1}{2}\pi - \varepsilon < y < \frac{1}{2}\pi + \varepsilon$.

Finally, it enters the rectangle

$$\pi - \varepsilon < x < \pi, \quad 0 < y < \int_0^\pi \frac{\sin x}{x} dx + \varepsilon,$$

and the procedure in the first region is repeated. *

THE GIBBS' PHENOMENON FOR THE FOURIER'S SERIES IN GENERAL.

§ 5. Let $f(x)$ be a function with an ordinary discontinuity when $x=a$, which can be expanded in a Fourier's Series in the interval $-\pi \leq x \leq \pi$.

Denote as usual by $f(a+0)$ and $f(a-0)$, the values towards which $f(x)$ tends as x approaches a from above or below. It will be convenient to consider $f(a+0)$ as greater than $f(a-0)$ in the description of the curve, but this restriction is in no way necessary.

Let $\phi(x-a) = 2 \sum_{r=1}^{\infty} \frac{1}{(2r-1)} \sin(2r-1)(x-a)$. Then

$$\left. \begin{aligned} \phi(x-a) &= \frac{1}{2}\pi, \text{ when } a < x < \pi+a, \\ \phi(x-a) &= -\frac{1}{2}\pi, \text{ when } -\pi+a < x < a, \\ \phi(+0) &= \frac{1}{2}\pi, \quad \phi(-0) = -\frac{1}{2}\pi, \\ \phi(0) &= 0 \text{ and } \phi(x) = \phi(x+2\pi). \end{aligned} \right\}$$

* The cosine series

$$\frac{\pi}{4} - \left[\cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x + \dots \right]$$

which represents 0 in the interval $0 \leq x < \frac{\pi}{2}$ and $\frac{\pi}{2}$ in the interval $\frac{\pi}{2} < x \leq \pi$ can be treated in the same way as the series discussed in this article.

Now put

$$\psi(x) = f(x) - \frac{1}{2} \{f(a+0) + f(a-0)\} - \frac{1}{\pi} \{f(a+0) - f(a-0)\} \phi(x-a),$$

and let $f(x)$, for $x=a$, be defined as $\frac{1}{2} \{f(a+0) + f(a-0)\}$.

Then $\psi(a+0) = \psi(a-0) = \psi(a) = 0$, and $\psi(x)$ is continuous at $x=a$.

The following distinct steps in the argument are numbered for the sake of clearness:

(i) Since $\psi(x)$ is continuous at $x=a$ and $\psi(a)=0$, if ϵ is a positive number, as small as we please, a number δ exists such that

$$|\psi(x)| < \frac{\epsilon}{4} \text{ for } |x-a| < \delta.$$

If δ is not originally less than ϵ , we can choose this part of δ for our interval.

(ii) $\psi(x)$ can be expanded in a Fourier's Series, this series being uniformly convergent in an interval $\alpha \leq x \leq \beta$ which includes no other discontinuity of $f(x)$ and $\phi(x-a)$ than $x=a$.

Let $s_n(x)$, $\phi_n(x-a)$ and $\sigma_n(x)$ be the sums of the terms up to and including those in $\sin nx$ and $\cos nx$ in the Fourier's Series for $f(x)$, $\phi(x-a)$ and $\psi(x)$. Then ϵ being the positive number of (i), as small as we please, there exists a positive integer ν' such that

$$|\sigma_n(x) - \psi(x)| < \frac{\epsilon}{4} \text{ for } n \leq \nu' \text{ in } \alpha \leq x \leq \beta.$$

Also

$$|\sigma_n(x)| \leq |\sigma_n(x) - \psi(x)| + |\psi(x)| < \frac{\epsilon}{4} + \frac{\epsilon}{4} < \frac{\epsilon}{2}$$

in $|x-a| < \delta$, if $\alpha < a - \delta < a < a + \delta < \beta$.

(iii) Now if n is even, the first maximum in $\phi_n(x-a)$ to the right of $x=a$ is at $a + \frac{\pi}{n}$; and if n is odd, it is at $a + \frac{\pi}{n+1}$. In either case there exists a positive integer ν'' such that the height of the first maximum lies between $\int_0^\pi \frac{\sin x}{x} dx$ and $\int_0^\pi \frac{\sin x}{x} dx + \frac{\pi\epsilon}{2\{f(a+0) - f(a-0)\}}$ for $n \geq \nu''$.

(iv) This first maximum will have its abscissa between a and $a + \delta$, provided that $\frac{\pi}{n} < \delta$.

Let ν''' be the first positive integer which satisfies this inequality.

(v) In the interval $a + \delta \leq x \leq \beta$, $s_n(x)$ converges uniformly to $f(x)$. Therefore, there exists a positive integer $\nu^{(iv)}$ such that for $n \geq \nu^{(iv)}$

$$|f(x) - s_n(x)| < \varepsilon \text{ in this interval.}$$

Now from the equation defining $\psi(x)$ we have

$$s_n(x) = \frac{1}{2}(f(a+0) + f(a-0)) + \frac{f(a+0) - f(a-0)}{\pi} \phi_n(x-a) + \sigma_n(x).$$

It follows from (i)-(v) that if ν is the first positive integer greater than ν' , ν'' , ν''' and $\nu^{(iv)}$, the curve $y = s_n(x)$ in the interval $a \leq x \leq \beta$, behaves as follows:

When $x=a$, it passes through a point whose ordinate is within $\frac{\varepsilon}{2}$ of $\frac{1}{2}(f(a+0) + f(a-0))$, and ascends at a steep gradient to its first maximum, which is at a height within ε of

$$\frac{1}{2}\{f(a+0) + f(a-0)\} + \frac{f(a+0) - f(a-0)}{\pi} \int_0^\pi \frac{\sin x}{x} dx.$$

This may be written

$$f(a+0) - \frac{f(a+0) - f(a-0)}{\pi} \int_\pi^\infty \frac{\sin x}{x} dx,$$

and, from Bôcher's table, referred to in § 3, we have

$$\int_\pi^\infty \frac{\sin x}{x} dx = -0.2811.$$

It then oscillates about $y=f(x)$ from $x=a$ to $a+\delta$, the character of the waves being determined by the function $\phi_n(x-a)$, since the term $\sigma_n(x)$ only adds a quantity less than $\frac{\varepsilon}{2}$ to the ordinate.

And on passing beyond $x=a+\delta$, the curve enters and remains within, the strip of width 2ε enclosing $y=f(x)$ from $x=a+\delta$ to $x=\beta$.

On the other side of the point a a similar set of circumstances can be established.

APPENDIX.

The following diagrams are the approximation curves

$$y = S_n(x)$$

of § 3 above, for $n=1, 2, 3$ and 4 . They illustrate the results (I-VI) of that section.

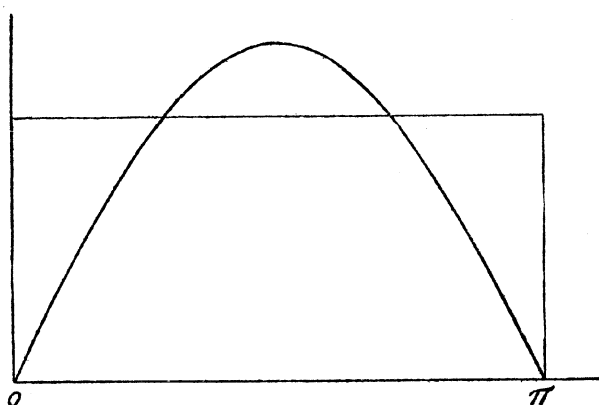


FIG. 1.

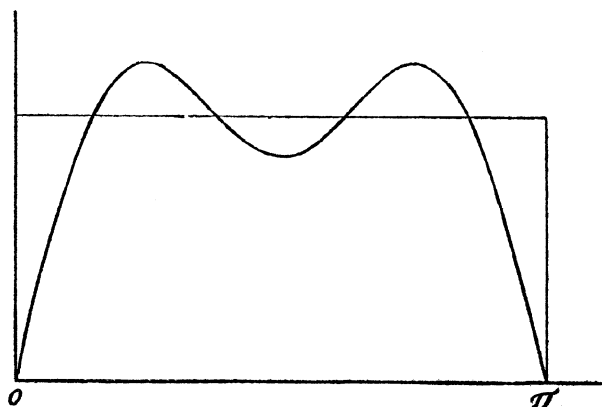


FIG. 2.

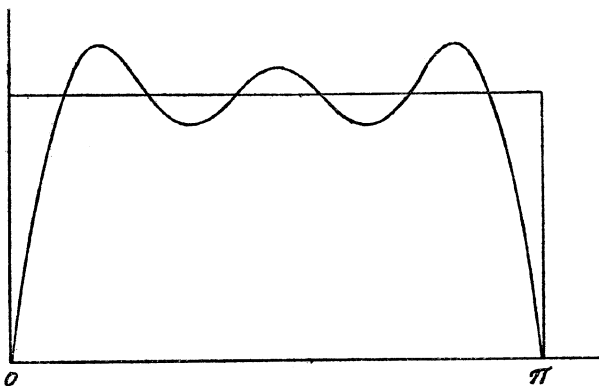


FIG. 3.

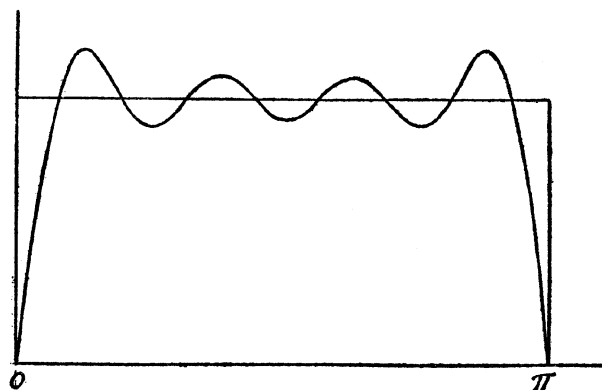


FIG. 4.